DATA ANALYSIS

Arithmetic Versus Geometric Means for Environmental Concentration Data

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Concentrations of chemical substances and microorganisms are often averaged for scientific and regulatory purposes. Geometric means are sometimes used for these purposes, but they are biased low and do not represent components of mass balances properly. They should be abandoned in favor of arithmetic means, unless they are clearly shown to be preferable for specific applications. Arithmetic means are unbiased, easier to calculate and understand, scientifically more meaningful (at least for concentration data), and more protective of public health. The bias and the root-mean-square error of the arithmetic mean, the geometric mean, and two bias-corrected geometric means are studied via simulation for several statistical distributions and sample sizes. The results demonstrate the general superiority of the arithmetic mean for concentration data, but a few exceptions are identified.

Environmental scientists routinely gather data about concentrations of chemical substances or microorganisms in air, water, and soil. Such data are often averaged for regulatory purposes and for estimating components of mass balances. Reported averages are sometimes expressed as geometric means rather than as more natural arithmetic means (*1–7*). The geometric mean is the antilog of the mean logarithm of a set of numbers or, equivalently, the *n*th root of the product of *n* numbers. (The latter definition remains valid even if some of the numbers are zero.) Why one should work with the antilog of the mean log of a set of measurements is unclear—there is no "law of conservation of logarithms of mass." When calculating the average value of a number of concentration measurements over space or time to represent some component of a mass balance, it makes more sense to sum the masses and then to divide by the sum of volumes (or by the total mass of matrix).

Geometric means may be useful for representing the average of a series of values that are always multiplied. For example, the average efficiency for a sequence of five processes involved in transforming the energy stored in underground oil to electrical energy in the home can be calculated as the one-fifth root of the product of the five process efficiencies. If that value were multiplied by itself five times, it would yield the overall efficiency (*cf. 8*). The arithmetic mean would not be useful for applications of this specialized type, but such applications seem rare. Geometric means may also be appropriate for averaging ratios and for rate constants in exponential processes (*9*).

The biased nature of the geometric mean

The bias inherent in geometric means, known since at least 1941, is the fundamental reason not to use it as a representative of the true population mean (*10*). Consider two streams with equal discharge Q that meet to form a larger stream with discharge $2Q$ (Figure 1). Stream A carries a concentration C_1 = 1 milligram (mg) liter $(L)^{-1}$ of total sulfur, whereas Stream B carries a concentration C_2 = 100 mg L^{-1}. After the two tributaries meet and mix, the resulting Stream D has a sulfur concentration C_3 = 50.5 mg L^{-1}, which is the arithmetic mean of the contributing concentrations. It is also the mass balance obtained using the flow-weighted average concentration $(Q_1C_1 + Q_2C_2)/(Q_1 + Q_2)$. The geometric mean here is the square root of the product of the two stream concentrations, or 10 mg L^{-1}, and does not represent a mass balance that would be useful to environmental scientists. Here, the difference between the two types of means is large be-

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FIGURE 1

Concentrations at a stream confluence

Streams A and B, each with discharge Q, carry concentrations of $C_1 = 1$ mg L^{-1} and $C_2 = 100$ mg L^{-1} of total sulfur, respectively. When they converge, they form stream D with discharge $2Q$. The resulting sulfur concentration is the arithmetic mean of C_1 and C_2 (50.5 mg L^{-1}), not the geometric mean (10 mg L^{-1}).

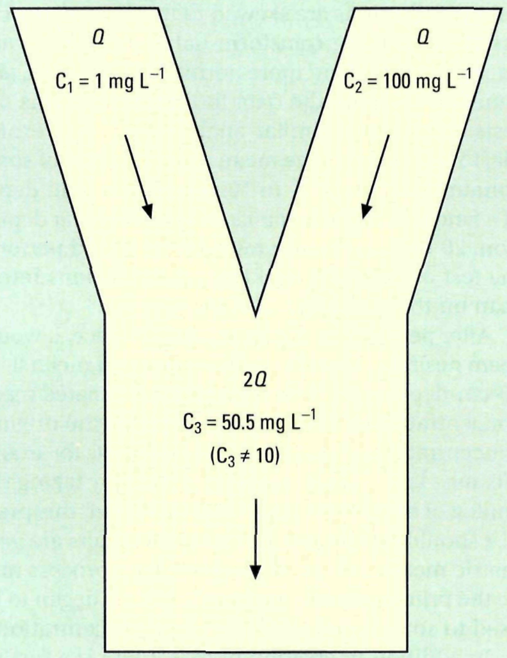
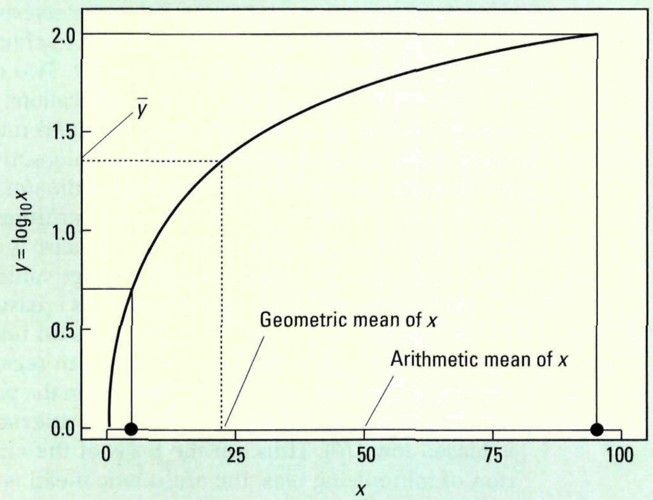


FIGURE 2

The source of the geometric mean's bias

The curvature of the logarithmic function causes large values to contribute relatively less to the geometric mean than to the arithmetic mean. This is the source of bias in a geometric mean (the antilog of the mean log) relative to the associated arithmetic mean. The solid dots at $x = 5$ and 95 are the "data."



cause of the 100-to-1 range in concentrations; the 80% error of the geometric mean is clearly undesirable.

Similarly, imagine taking 100 soil samples of equal mass from a plot of land contaminated with polychlorobiphenyls (PCBs). The best estimate of the average mass of PCBs per unit mass of soil in this landfill is the arithmetic mean concentration, which is equivalent to the result obtained by compositing all 100 samples, mixing well, and measuring the concentration of the mixture. Again, it is difficult to see the relevance of the geometric mean of concentrations for using data like these in a mass balance analysis.

Suppose in the landfill example that the sample PCB concentrations have a log-normal distribution (the logarithms of the concentrations have a normal distribution). It is common statistical practice to transform such sample concentrations to logs before estimating confidence limits or performing statistical tests, such as analysis of variance (anova) or t tests. However, if a mean of logarithms were then back-transformed to the original units by taking its antilog, the result would be a geometric mean, and it would be biased low relative to the result obtained by compositing samples, mixing well, and measuring the resulting concentration. This latter result, which is what is usually intended in mass balance calculations, is provided directly by the arithmetic mean of the original data.

There are no obvious situations in which the geometric mean is scientifically preferable to the arithmetic mean for representing the central tendency of concentration data for use in a mass balance or other additive analyses. The geometric mean is always smaller than the arithmetic mean unless all numbers in a dataset are identical, in which case the two will be equal. This bias results from the curvature of the logarithmic function (Figure 2), which downplays larger values relative to smaller ones. In the figure, the antilog (21.8) of the mean of the logarithms of two numbers is smaller than the ordinary mean (50.0). Because of their bias, geometric means are of little use for mass balance analyses or for fate and transport studies of either chemical substances or microorganisms in the environment. Furthermore, geometric means should not be used for regulatory purposes; because of their bias, they are less protective of public health than are arithmetic means. These observations raise the question: Why have geometric means become popular?

Use of geometric means is questionable

Available literature does not provide justifications for using geometric means. Although one report does suggest that "the best estimate of central tendency of log-normal data is the geometric mean" (*11*), no definition of "best" and no reference providing further justification accompany that statement.

Is the geometric mean a robust estimator of the true population mean? Distributions of environmental concentrations are often skewed to the right and may be log-normally distributed. In such situations, when low values are common, and high values are rare, occasional samples drawn from those distributions will contain one or more very large, but real, values. These samples can have unusually large arithmetic means.

When individual data come from a strongly skewed distribution, means estimated from a series of small samples (e.g., $N \approx 10$) will also have a skewed distribution, although less skewed than that of the original data. This suggests a need to examine how well sample means represent the true mean of the population from which the samples are drawn.

In the statistical theory that deals with estimation of parameters, such as population means based on sample data, several criteria are used to choose among various candidate estimators (*12*). Two criteria of importance in environmental applications are minimum bias (a measure of accuracy) and minimum mean-squared error (a combined measure of accuracy and precision). The bias of an estimator (in this case, an estimator of the population arithmetic mean, which is appropriate for mass balances) is defined as the difference between the average value of the estimator and the population mean. It is easy to prove that the arithmetic mean provides an unbiased estimate of the true population mean regardless of the distribution of individual items in the population (*13*). It is also known that the geometric mean is biased low (*10*). Thus, on the basis of the criterion of minimizing bias, the arithmetic mean is always preferable to the geometric mean.

Bias is not the only factor to consider in determining the value of an estimator. For example, consider a population with a true mean of 100. Suppose some bizarre estimator or measurement system yielded sample means of 0 half of the time and 200 half of the time. The estimator would be correct on average and would be unbiased. However, an alternative estimator that always gave values between 90 and 100 would usually be preferable, even if its average value were 95, making it biased. This alternative estimator would be closer on average to the true mean in an absolute sense. The mean-squared error (the average squared difference between potential estimates and the true mean) can be used to consider this feature of estimators because it accounts for bias and variability simultaneously. The geometric mean might therefore be preferred as an estimator, at least for some situations, if it provided estimates with lower mean-squared error than the arithmetic mean; but, as will be seen, it does not do so for most cases considered in the described simulations.

In following sections, the square root of the mean-squared error (the root-mean-squared error, RMSE) is used in preference to the mean-squared error, because the former metric has the same units as the original concentrations and thus has more commonsense meaning than the mean-squared error. For an unbiased estimator, the RMSE is identical to the standard error; for a biased estimator, it is larger because of the bias.

Are logarithmic transforms needed for statistical analysis, and if so, should these lead to use of geometric means? Many environmental variables probably have distributions that are closer to lognormal than to normal, and although empirical distributions can be skewed to the left, many more are skewed to the right (*14*). Koch provided reasons why this may be expected theoretically (*15, 16*). Related to log normality, it is common for the standard deviations of different subsets of data to be roughly proportional to, or at least increase with, the means of those subsets. When these phenomena occur, working with the logarithms of the original data tends to make the standard deviations more uniform and independent of the means and makes the data more normally distributed.

Within this framework, suppose that two or more groups of data are to be compared statistically. When data distributions are skewed to the right, researchers frequently log transform data to stabilize variances, to make them more normally distributed, and generally to make the data fit the assumptions of t tests, anova, and similar analyses (*17*). For example, to test whether the mean concentration of some contaminant in the 0- to 10-centimeter (cm) depth of a landfill is higher than its concentration at depths from 20 to 30 cm, some researchers would perform the test on logarithms of the concentrations rather than on the concentrations themselves.

After performing such a test in log space, it would seem desirable to indicate the estimated mean 0- to 10-cm-depth concentration and the estimated mean concentration at 20- to 30-cm depth in the original concentration units of the contaminant, for example, $mg \cdot kg^{-1}$. This is sometimes done by taking the antilog of each of the mean logs. However, this practice should be avoided, because the results are geometric means and are biased low. This process may be the primary reason geometric means began to be used to summarize environmental concentrations.

In addition to creation of bias, there is a further problem with logarithmic transformation of data. Environmental scientists would usually be interested in comparisons of arithmetic means and not whether the mean logarithm of one group is larger than that of another group. Unfortunately, it is possible for the arithmetic mean of one group, for example $A = (10, 90)$, to be larger than that of another group, for example $B = (40, 50)$, but for the mean of the logarithms of B (1.651) to be greater than the mean of the logarithms of A (1.477) (*18, 19*). A test based on log transforms may, therefore, be irrelevant or even misleading to the scientific question at hand. As noted by one reviewer, other kinds of statistics—permutation tests or likelihood-ratio tests—should replace t tests or anova for such data for this reason (*13, 20, 21*).

Does the geometric mean represent commonly occurring values? When samples are taken from populations having log-normal or similarly skewed distributions, occasional samples will contain high values, because occasional high values are naturally part of such distributions. Those samples that do contain the high values will often have means that are larger than the true population mean, and such means can also be larger than values that occur commonly. Note that with arithmetic means, these samples will be balanced on average by samples having means that are lower than the population mean.

Reckhow and Chapra have stated that "the 'population' statistics that best represent the center of the population of lake phosphorus concentration values are probably the weighted geometric mean and the median" (2). One colleague has offered an opinion that geometric means are "more representative" of commonly occurring values than are arithmetic means and should be used for that reason. It is true that for log-normally distributed variables, the geometric mean is equal on average to the population median—to the value that is exceeded half the time. However, for data having other distributions, this relationship does not hold, at least not exactly. If mass balances are of interest, medians will underestimate population means for any distribution that is skewed to the right, and arithmetic means are a more appropriate metric. This point is closely connected to the following one.

Does the geometric mean downplay large values? Michels has stated that the naive bias-corrected geometric mean (discussed below) "is preferred for estimating [the arithmetic mean of concentrations of many contaminants] since it is less sensitive to uncertainties in a few high values" (5). I have also heard the opinion that concentrations of coliform bacteria in swimming or drinking water should be averaged as geometric means rather than arithmetic means to downplay the large values that are sometimes measured. Those making such a suggestion seem to believe that the large values result from measurement error rather than from natural variability in the population of concentrations being sampled. Yet either explanation is possible, and if concentrations actually do vary according to a log-normal distribution, then occasional large values will occur naturally, and neglecting them will lead to an underestimated true mean.

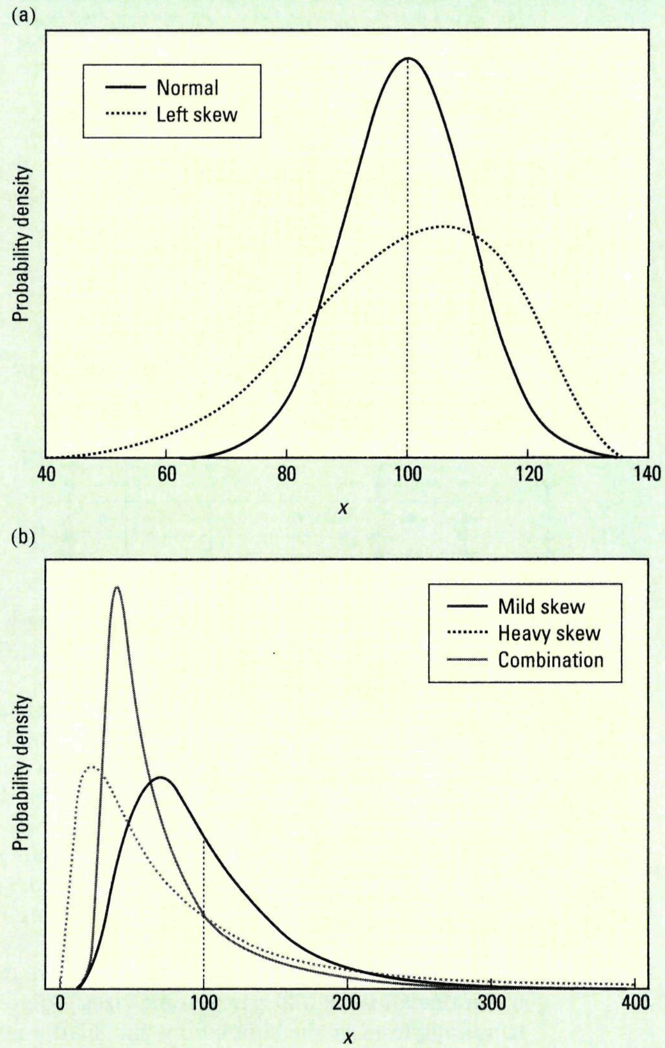
It is true that the processes used to estimate bacterial concentrations in water are subject to sampling errors. However, there is no reason to suppose that such errors are biased high. Current standard methods for analyzing bacterial concentrations can be expected to give values that are correct on average—for every result that is too high, there should be counterbalancing results that are too low. This may take the form of several values that are a little too low balancing one value that is much too high. There is no clear justification for downplaying high values—it is undesirable to use estimators of means that are biased low when seeking to protect public health from dangerous organisms or chemicals.

Should geometric means be used for characterizing populations of microorganisms that reproduce geometrically? It is difficult to see the relevance of the reproduction mode to the average number or concentration of organisms present. For example, if bacterial concentration in a water body increased in the sequence 100, 200, 400, 800, 1600, 3200, 6400 over a seven-day period, the mean concentration for that week would be about 1814.3 (the arithmetic mean), not 800 (the geometric mean). Moreover, geometric reproduction occurs only in unlimited environments. In most real situations, other patterns of net increase occur. As early as 1955, Thomas demonstrated the advantages of arithmetic over geometric means for concentrations of coliform bacteria (22).

FIGURE 3

Probability density curves

Curves for the distributions used in the simulations include: (a) normal distribution and left-skewed beta distribution, (b) mildly skewed and heavily skewed log-normal distributions, and a distribution that is a 50:50 combination of the normal distribution and the heavily skewed log-normal distribution. The vertical dashed lines mark the population mean ($\mu = 100$) of all five distributions.



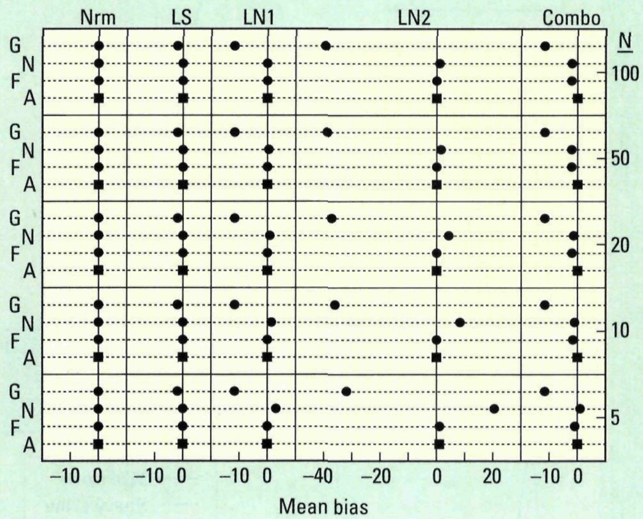
Simulation study

When environmental concentration data are averaged, a common goal is to estimate the true (population) arithmetic mean, which is appropriate for use in mass balances. Although the sample arithmetic mean is unbiased, it may not provide minimum RMSE for non-normal distributions. Therefore, I compared its performance with that of the geometric mean and two bias-corrected geometric means to see whether they might provide an improved compromise between bias and RMSE. One of the latter two estimators, a naive bias-corrected geometric mean (NBCGM), invokes a naive bias correction (23) that can be far from exact when only a sample mean and variance are available. The Finney bias-corrected geometric mean (FBCGM) conjures a

FIGURE 4

Estimator bias

Mean bias of four estimators (left) of the population arithmetic mean, based on 25,000 samples of size *N* (right) drawn from five statistical distributions (top): normal distribution (Nrm), left-skewed beta distribution (LS), mildly skewed (LN1) and heavily skewed (LN2) log-normal distributions, and a distribution that is a 50:50 combination (Combo). Estimators: G, geometric mean; N, naive bias-corrected geometric mean; F, Finney bias-corrected geometric mean (FBCGM); A, arithmetic mean. Zero bias is most desirable. Square points emphasize that the sample arithmetic mean has a lower bias than the geometric mean for every combination of distribution and sample size. Although not easily seen, the FBCGM has slightly lower bias than the arithmetic mean for six combinations.



correction proposed by Finney that is an unbiased estimator of population means for log-normal distributions and, of all such estimators, has the minimum variance (*10*, *24*). These corrections are defined and described in Supporting Information.

To check the sampling properties of these four estimators, numerical simulations were performed. The arithmetic mean, both bias-corrected geometric means, and the uncorrected geometric mean were calculated for each combination of five different distributions and five different sample sizes. The distributions used in the simulations included a normal distribution, a left-skewed distribution, a mildly skewed log-normal distribution, a heavily skewed log-normal distribution, and a distribution that combined the normal and the heavily skewed log-normal distributions in equal proportions (Figure 3a and 3b). The five sample sizes considered were $N =$ 5, 10, 20, 50, and 100. For these calculations, 25,000 samples of a given sample size, for example $N = 5$, were drawn from each specified distribution. Finally, for each 25,000-sample simulation, the mean bias and the RMSE were calculated for each of the four estimators.

These simulations determined the amounts of bias and RMSE associated with each of the four estimators for each distribution and sample size. The measures of error could be computed directly using numerical integration for uncorrected geometric means of the two log-normal distributions (*25*) but not for the other combinations. For this reason, the results discussed are those obtained from the numerical simulations, but possible comparisons with theoretical results will be discussed.

Comparisons of estimator behavior

The mean biases, averaged over 25,000 samples for each of the possible combinations, are shown in Figure 4. The scale factor for the horizontal axis is identical for all five distributions. The following features stand out:

- The smoothness of (imaginary) curves connecting the points for a given combination of estimator and distribution demonstrates that 25,000 simulations were adequate to produce stable estimates. For the geometric means of the two log-normal distributions, the simulation results were all within 1.5% of the values obtained theoretically.

The arithmetic means have lower absolute values of bias than the geometric means for all 25 combinations of distribution and sample size. The FBCGM had lower absolute bias than the arithmetic mean for six of the combinations, but only by very small amounts that are within the uncertainty of the simulations. In additional simulations based on a highly skewed Weibull distribution (see Supporting Information), the bias of the FBCGM ranged from 177 to 554. This estimator may thus behave poorly for skewed data that are not log normal. Because arithmetic means are unbiased for all distributions, and empirical data are unlikely to come from pure log-normal distributions, the arithmetic mean seems preferable even to the FBCGM in most situations.

- For normal and left-skewed distributions, there is little bias with any of the estimators at any sample size, but for the three right-skewed distributions, the negative bias of the uncorrected geometric mean is easily observed (Figure 4). For the heavily skewed distribution, bias is substantial.

The NBCGM is known to overcorrect for the negative bias of the geometric mean, leading to positive bias (*24*, *26*). This occurs especially at small sample sizes (Figure 4), probably because the sample variance is a poorer estimate of the population variance in those cases than when sample size is larger.

Overall, the arithmetic mean is the clear choice over uncorrected geometric means to minimize bias. Furthermore, the bias-corrected geometric means are never better than the arithmetic means by enough to compensate for their greater computational complexity.

Landwehr has commented on problems associated with using geometric means for regulating water quality, based on an analytic rather than simulation-based analysis (*27*). For all statistical distributions he investigated, larger sample sizes led to smaller average geometric means; this same effect can be seen for all five distributions analyzed here (Figure 4).

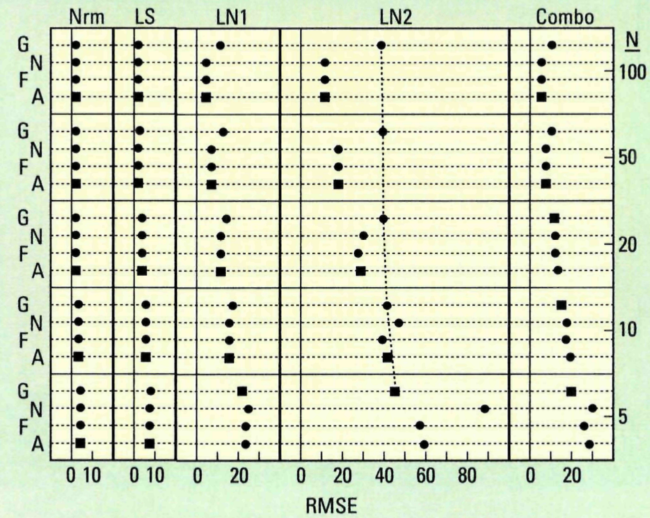
A biased estimator might sometimes be preferred to an unbiased one if the former were sufficiently closer to the true value being estimated. The RMSE is a useful measure of the average distance between an estimator and its target (Figure 5). Values of RMSE obtained in the simulations reveal the following:

- For a given combination of distribution and sample size within each "box" (Figure 5), the square

FIGURE 5

Estimator root-mean-square error

Root-mean-square error (RMSE) of four estimators (notations as in Figure 4) of the population arithmetic mean, based on 25,000 samples of size N (right) drawn from five statistical distributions (top): normal distribution (Nrm), left-skewed beta distribution (LS), mildly skewed (LN1) and heavily skewed (LN2) log-normal distributions, and a distribution that is a 50:50 combination (Combo). Zero RMSE is most desirable. Square points indicate whether the arithmetic mean or the uncorrected geometric mean has a lower mean error. The dashed line emphasizes the performance of the uncorrected geometric mean with the highly skewed log-normal distribution.



point is assigned to the arithmetic or uncorrected geometric means, whichever has the lower average error. The geometric mean has a lower error than the arithmetic mean in five boxes at the lower right corner of the figure, where for the heavily skewed log-normal distribution at $N = 5$, its RMSE is about 23% lower than that of the arithmetic mean.

For the combination distribution at $N = 5$, this increases to a 28% difference. The geometric mean has slightly lower RMSE than the arithmetic mean for this distribution for samples of 10 and 20 as well, but it is biased low by about 10 units in those cases. This advantage of the geometric mean is swamped by its poor performance at larger sample sizes. For example, for the heavily skewed log-normal distribution, and $N = 100$, the error of the geometric mean is about threefold greater than that of the arithmetic mean.

- In general, the RMSE of all four estimators declines with increasing sample size within each distribution. The decline is, however, very slight for the geometric mean with the heavily skewed log-normal distribution.
- The FBCGM has a marginally lower RMSE (2.3 units maximum) than the arithmetic mean for the heavily skewed log-normal and combination distributions when sample size is 20 or less, and it never has substantially greater error for the distributions plotted. However, its RMSE ranged from 517 to 19,361 for the simulation based on the Weibull distribution (not shown). Because of the complexity of this correction and its possible inaccuracy for distributions that are not log normal, the simpler arithmetic mean again seems more advantageous for most uses.

Some of these properties are illustrated with means of 100 simulated samples (Figure 6) in which each point represents one mean of either $N = 5$ or $N = 100$. These samples were drawn from the heavily skewed log-normal distribution, with its population mean of 100 units. For very small samples ($N = 5$), the arithmetic sample means can depart far from the true mean so that geometric means might be preferable despite their greater bias. However, for the larger samples ($N = 100$), the geometric means are tightly clustered around a highly biased value, so their average error (measured as RMSE) is much greater than the average error of the arithmetic means.

Taking both criteria—minimum bias and minimum RMSE—into account, one can nearly eliminate the uncorrected geometric mean and the NBCGM from general consideration, leaving the arithmetic mean and the FBCGM from which to choose as practical estimators of true population means. However, note that the arithmetic mean has a bias closer to zero than the FBCGM for most combinations of sample size and distribution tested; the FBCGM improves very little on the RMSE estimate compared with the arithmetic mean in the cases tested where it is smaller; the FBCGM can have substantial bias and error for at least some skewed distributions; and arithmetic means are much easier to calculate and interpret than are bias-corrected geometric means (see Supporting Information). Thus, the arithmetic mean seems the best overall choice for most sample sizes and distributions considered here.

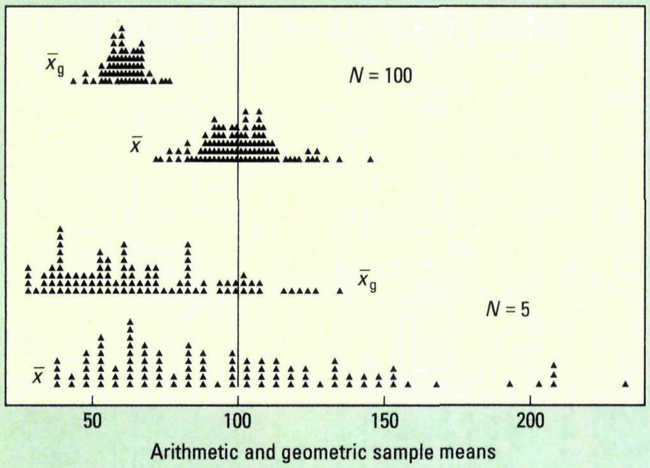
Only if very small samples, for example, $N = 5$, are consistently taken from distributions known to be heavily skewed and log normal does the uncorrected geometric mean appear to be the estimator of choice (and then only if its reduced RMSE is considered to compensate for its high bias). This contrasts with its current common use with much larger samples and with distributions that may not be so heavily skewed or may not be log normal. Although one might expect the geometric mean to perform best for a heavily skewed log-normal distribution, the raw geometric mean actually performs worst for that distribution, except for the minimum-RMSE criterion at very small sample sizes.

In sum, for all but a few particular cases, arithmetic means have many advantages over geometric means, even when bias corrections are applied. Geometric means might reasonably be used only in situations where means of consistently small samples are needed for data drawn from log-normal distributions. Somewhat paradoxically, however, large samples are required to determine the shapes of distributions with any confidence (*18*). Haas recently recommended the use of arithmetic, not geometric, means for risk analyses based on microbial densities for reasons similar to those expressed here (*28*). In cases where they have, for whatever reason, become established as the standard way to summarize data, geometric means might continue to be reported along with arithmetic means to allow com-

FIGURE 6

Comparative advantages of two estimators of means

The arithmetic and geometric sample means for samples of size $N = 5$ (lower pair) and $N = 100$ (upper pair) drawn from the heavily skewed log-normal distribution, LN2. Each dot plot represents 100 sample means. At $N = 5$, the higher average precision of the geometric means $\bar{x}_g$ might compensate for their lower accuracy. At $N = 100$, the arithmetic means $\bar{x}$ are much less biased but nearly as precise as the geometric mean. The vertical line represents the target—the population mean, $\mu = 100$.



parisons over time. However, they should not be used in mass-balance work and should be phased out as regulatory criteria as soon as it is practical. At a minimum, if geometric means are to be used for some reason, they should always be bias-corrected by the Finney method.

Supporting information

Ancillary material, including a detailed description of the "naive" and Finney bias corrections for the geometric mean, details of the distributions used in the simulations, a warning about the likelihood of finding so-called outliers that are proper components of log-normal distributions, and information on confidence limits for log-normal data, will appear following these pages in the microfilm edition of this journal volume. Supporting information is available to subscribers electronically via the World Wide Web at http:// pubs.acs.org and via Gopher at pubs.acs.org. Photocopies of the supporting information from this paper or microfiche (105 × 148 mm, 24 × reduction, negatives) may be obtained from the Microforms Office, American Chemical Society, 1155 16th St., NW, Washington, D.C. 20036. Full bibliographic citation (journal, number, and issue number) and prepayment, check or money order for $25.50 for photocopy ($27.50 foreign) or $12.00 for microfiche ($13.00 foreign), are required. Canadian residents should add 7% GST.

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References

(1) Larsen, R. I. *A Mathematical Model Relating Air Quality Measurements to Air Quality Standards*; Air Programs Report No. AP-89; U.S. Environmental Protection Agency, U.S. Government Printing Office: Washington, DC, 1971.
(2) Reckhow, K. H.; Chapra, S. C. *Engineering Approaches for Lake Management. Volume 1: Data Analysis and Empirical Modeling*; Butterworth Publishers: Boston, MA, 1983.
(3) Brzuzy, L. P.; Hites, R. A. *Environ. Sci. Technol.* **1995**, *29*, 2090.
(4) St. Louis, V. L.; Rudd, J. W. M.; Kelly, C. A.; Beaty, K. G.; Flett, R. J.; Roulet, N. T. *Environ. Sci. Technol.* **1996**, *30*, 2719.
(5) Michels, D. E. *Environ. Sci. Technol.* **1977**, *11*, 300.
(6) *Quality Criteria for Water 1986*; EPA 440/5-86-001, U.S. Environmental Protection Agency, Office of Water Regulations and Standards, U.S. Government Printing Office: Washington, DC, 1986.
(7) *DMR Manual for Completing the Discharge Monitoring Report for the National Pollution Discharge Elimination System (NPDES)*; N.Y. State Department of Environmental Protection, Division of Water: Albany, NY, 1994.
(8) Ricklefs, R. E. *Ecology*, 3rd ed.; W. H. Freeman: New York, 1990.
(9) Croxton, F. E.; Cowden, D. J.; Klein, S. *Applied General Statistics*, 3rd ed.; Prentice-Hall: Englewood Cliffs, NJ, 1967.
(10) Finney, D. J. *J. Roy. Stat. Soc. (Suppl.)* **1941**, *7*, 155.
(11) *Standard Methods for the Examination of Water and Wastewater*, 18th ed.; Greenberg, A. E., Clesceri, L. S., Eaton, A. D., Eds.; American Public Health Association: Washington, DC, 1992.
(12) Greene, W. H. *Econometric Analysis*; Macmillian Publishing: New York, 1993.
(13) Freund, J. E.; Walpole, R. E. *Mathematical Statistics*, 4th ed.; Prentice-Hall: Englewood Cliffs, NJ, 1987.
(14) Smith, D. G.; Maasdam, R. *N. Z. J. Mar. Freshwater Res.* **1994**, *28*, 19.
(15) Koch, A. L. *J. Theor. Biol.* **1966**, *12*, 276.
(16) Koch, A. L. *J. Theor. Biol.* **1969**, *23*, 251.
(17) Box, G. E. P.; Hunter, W. G.; Hunter, J. S. *Statistics for Experimenters*; Wiley & Sons: New York, 1978.
(18) Fisher, L. D.; van Belle, G. *Biostatistics: A Methodology for the Health Sciences*; Wiley & Sons: New York, 1993.
(19) Zhou, X.-H.; Gao, S.; Hui, S. L. *Biometrics* **1997**, *53*(3), 1129.
(20) Good, P. *Permutation Tests*; Springer-Verlag: New York, 1994.
(21) Lindsey, J. K. *Parametric Statistical Inference*; Oxford University Press: New York, 1996.
(22) Thomas, H. A., Jr. *Sewage Ind. Wastes* **1955**, *27*, 212.
(23) Miller, D. *Am. Stat.* **1984**, *38*, 124.
(24) Gilbert, R. O. *Statistical Methods for Environmental Pollution Monitoring*; Van Nostrand Reinhold: New York, 1987.
(25) Aitchison, J.; Brown, J. A. C. *The Lognormal Distribution with Special Reference to Its Uses in Economics*; Cambridge University Press: Cambridge, U.K., 1957.
(26) Kendall, M. G.; Stuart, A. *The Advanced Theory of Statistics*; Charles Griffin: London, 1961; Vol. 2.
(27) Landwehr, J. M. *Water Resour. Res.* **1978**, *14*, 467.
(28) Haas, C. N. *Water Res.* **1996**, *30*, 1036.
(29) Cleveland, W. S. *Visualizing Data*; Hobart Press: Summit, NJ, 1993.